Optimal condition of income a company modeled discrete equations with Markovian coefficients

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Abstract

The paper deals with a system of difference equations where coefficients depend on Markov chains. The functional equations for particular density and the moment equations for the system are derived and used in the investigation of mode stability of income a company. An application of the results is supported by two models.

1 Introduction

Already at the beginning of the 20th century it was found that even in a sequence of equally distributed independent random variables could occur quite naturally marginal distributions other than the normal. Mechanism of creation majority of such regularities can be understood only using the

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theory of the Markov chains or Markov processes. Process in a system is called a Markov chain, if it undergoes transitions from one state to another while the next state depends only on the current state and not on the sequence of events that preceded it. The number of possible states is finite or countable. Based on the stochastic approach, we can study a number of aspects concerning a variety of different phenomena in finance and economics. Development of a company profit models is also possible within the framework of the stochastic approach using a Markov chain. The first financial model to use a Markov chain was until the end of the 20th century.

In this paper, we offer to study constructions of mathematical models of complex economic systems that belong to the class of systems operating in conditions of uncertainty. Incomplete information, their distortion, lack of observations, changing structure over time, the stochastic nature of the impact of the external environment and others, all of this creates uncertainty conditions in which the system works. Problems of such mathematical models arise from the fact that the data type "input - output" are noisily and nonlinearly. Specifically, in our models as a modifier appears "white noise".

In the focus of our attention is construction of moment equations for determining the expected value of the guaranteed profit of a company. The theoretical results are applied on two models of income a company.

2 Moment Equations

Let $(\Omega, \mathcal{F}, F, \mathbb{P})$ be a filtered probability space (or stochastic basis) consisting of a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration

$$F = \{\mathcal{F}_t, \ \forall t \geq 0\} \subset \mathcal{F},$$

(for definitions see, e.g. [10]). The space Ω is called sample space, $\mathcal{F}$ is thea set of all possible events (a σ-algebra), that may occur to the moment t, and $\mathbb{P}$ is some probability measure on Ω. Such random space plays fundamental role in the construction of models in economics, finance etc.

On the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, we consider initial problem formulated for the stochastic dynamic system with random coefficients in the form

$$x_{n+1} = A(\xi_{n+1}, \xi_n)x_n + B(\xi_{n+1}, \xi_n), \quad n = 1, 2, \dots \tag{1}$$

$$x_0 = \varphi(\omega), \qquad \varphi \colon \Omega \to \mathbb{R}^m \tag{2}$$

where A is $m \times m$ matrix with random elements, ξ_n is the Markov chain of finite number of the states $\theta_1, \theta_2, \dots, \theta_q$ with the probabilities $p_k(n) = P\{\xi_n = \theta_k\}$, $k = 1, 2, \dots, q$, $n = 1, 2, \dots$ that satisfy the system of difference equations

$$p_k(n+1) = \sum_{s=1}^{q} \pi_{ks} p_s(n), \quad k = 1, 2, \dots, q \tag{3}$$

where $\left(\pi_{ks}(t)\right)_{k,s=1}^{q}$ is transition matrix. This transition matrix we denote $\Pi = \left(\pi_{ks}(t)\right)_{k,s=1}^{q}$.

If the random variable ξ_{n+1} is in state θ_k, $k = 1, 2, \dots, q$ and the random variable ξ_n is in state θ_s, $s = 1, 2, \dots, q$, for simplicity we denote

$$A_{ks} = A(\theta_k, \theta_s), \quad B_{ks} = B(\theta_k, \theta_s), \quad k, s = 1, 2, \dots, q.$$

The state m-dimensional column vector-function x_n, $n = 1, 2, \dots$ is called a solution of the system (1) if x_n, $n = 1, 2, \dots$ satisfies (1) within the meaning of a strong solution [6].

Our task is to derive the moment equations of system (1) which will be used for determining the mode stability of income a firm.

We define the moments of the first or second order of a solution x_n, $n = 1, 2, \dots$ of (1) before we derive the moment equations. We use some notation. In the sequel, $\mathbb{E}_m$ denotes an m-dimensional Euclidean space, functions $f_k(n, x)$, $n = 1, 2, \dots, k = 1, 2, \dots, q$ are the particular density functions of x_n, $n = 1, 2, \dots$ determined by the formula (see in [])

$$\int_{\mathbb{E}_m} f_k(n, x)\, dx = P\big\{x_n \in \mathbb{E}_m, \xi_n = \theta_k\big\}, \qquad k = 1, 2, \dots, q \tag{4}$$

and satisfy the following functional equations

$$f_k(n+1, x) = \sum_{s=1}^{q} \pi_{ks} f_s\big(n, A_{ks}^{-1}(x - B_{ks})\big) \det A_{ks}^{-1}, \quad k = 1, 2, \dots, q \tag{5}$$

Definition 1. *The vector function*

$$E^{(1)}\{x_n\} = \sum_{k=1}^{q} E_k^{(1)}\{x_n\}$$

where

$$E_k^{(1)}\{x_n\} = \int\limits_{\mathbb{E}_m} x f_k(n,\, x)\, dx, \qquad k = 1, 2, \ldots, q, \tag{6}$$

is called moment of the first order for a solution x_n, $n = 1, 2, \ldots$ of (1). The values $E_k^{(1)}\{x_n\}$ are called particular moments of the first order.

Definition 2. *The matrix function*

$$E^{(2)}\{x_n\} = \sum_{k=1}^{q} E_k^{(2)}\{x_n\}$$

where

$$E_k^{(2)}\{x_n\} = \int\limits_{\mathbb{E}_m} x x^* f_k(n,\, x)\, dx, \qquad k = 1, 2, \ldots, q \tag{7}$$

is called moment of the second order for a solution x_n, $n = 1, 2, \ldots$ of (1). The values $E_k^{(2)}\{x_n\}$ are called particular moments of the second order.

Theorem 1. *Systems of moment equations of the first or the second order respectively for a solution x_n, $n = 1, 2, \ldots$ of (1) are of the form*

$$E_k^{(1)}\{x_{n+1}\} = \sum_{s=1}^{q} \pi_{ks} \Big(A_{ks} E_k^{(1)}\{x_n\} + B_{ks} p_s(n) \Big), \tag{8}$$

$$\begin{aligned} E_k^{(2)}\{x_{n+1}\} = \sum_{s=1}^{q} \pi_{ks} \Big(& A_{ks} E_k^{(2)}\{x_n\} A_{ks}^* + A_{ks} E_s^{(1)} B_{ks}^* \\ & + B_{ks} E_s^{(1)} A_{ks}^* + B_{ks} B_{ks}^* p_s(n) \Big). \end{aligned} \tag{9}$$

Proof. Multiplying equation (5) by x and integrating them on Euclidean space $\mathbb{E}_m$ we obtain the system

$$\int\limits_{\mathbb{E}_m} x f_k(n+1,\, x) dx = \sum_{s=1}^{q} \pi_{ks} \int\limits_{\mathbb{E}_m} x f_s\big(n,\, A_{ks}^{-1}(x - B_{ks})\big) \det A_{ks}^{-1} dx. \tag{10}$$

Using the substitution $Y_{ks} = A_{ks}^{-1}x$, integrating by parts, in regard to $\int_{\mathbb{E}_m} f_k(n+1,\, x)dx = p_s(n)$, we get systems of moment equations (8).

In the same way, it can be derived system of moment equations (9). That means, equation (5) is multiplied by xx^* and integrated by parts on Euclidean space $\mathbb{E}_m$.

□

3 Model problem 1

We consider a probability space $(\Omega,\, \mathcal{F},\, \mathbf{P})$ with filtration $\{\mathcal{F}_t\}$ as a space of trade relations. The changes in the level of income a company can be modelled by using stochastic difference equations. Such convenient mathematical model is the scalar case of the initial problem (1), (2) that is,

$$\begin{aligned} &x_{n+1} = x_n + b(\xi_{n+1},\, \xi_n), \quad n = 1, 2, \dots \\ &x_0 = \varphi(\omega). \end{aligned} \tag{11}$$

The stochastic equation (11) describes the graph of income a company with initial value of income x_0. Inhomogeneity $b(\xi_{n+1},\, \xi_n)$ here expresses the conditions in which the company works. For example, the value $b(\theta_k,\, \theta_s),\ k, s = 1, 2, \dots, q$ means the transition from one state θ_k of company activities, may be state of crisis, to another state θ_s, may be post-crisis situation, or the like.

We denote $\quad b(\theta_k,\, \theta_s) = b_{ks},\ k, s = 1, 2, \dots, q.$

Transition probabilities satisfy system (3), which in the scalar case takes the form

$$p(n+1) = \Pi\, p(n), \quad n = 1, 2, \dots. \tag{12}$$

Then, the moment equations (8), (9) take the form

$$E^{(1)}\{x_{n+1}\} = \Pi\, E^{(1)}\{x_n\} + B_1\, p(n), \tag{13}$$

$$E^{(2)}\{x_{n+1}\} = \Pi E^{(2)}\{x_n\} + 2B_2 E^{(1)}\{x_n\} + B_2\, p(n) \tag{14}$$

where

$$B_1 = \begin{pmatrix} \pi_{11}b_{11} & \pi_{12}b_{12} & \cdots & \pi_{1q}b_{1q} \\ \pi_{21}b_{21} & \pi_{22}b_{22} & \cdots & \pi_{2q}b_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{q1}b_{q1} & \pi_{q2}b_{q2} & \cdots & \pi_{qq}b_{qq} \end{pmatrix}, \quad B_2 = \begin{pmatrix} \pi_{11}b_{11}^2 & \pi_{12}b_{12}^2 & \cdots & \pi_{1q}b_{1q}^2 \\ \pi_{21}b_{21}^2 & \pi_{22}b_{22}^2 & \cdots & \pi_{2q}b_{2q}^2 \\ \vdots & \vdots & \ddots & \vdots \\ \pi_{q1}b_{q1}^2 & \pi_{q2}b_{q2}^2 & \cdots & \pi_{qq}b_{qq}^2 \end{pmatrix}.$$

The first or the second moments can be obtained solving systems of difference equations(12) and (13) or (12) and (14) respectively, for example by using a numerical method.

Example 1. The real expected mean value of income can be determined in a particular case. Assume that a company is trying to get out of the crisis, thus gets into three possible states corresponding to the three possible manipulations, those are:

1. in a salary adjustment,
2. in an amount of employees,
3. by credits.

Let the transition matrix have the form

$$\Pi = \begin{pmatrix} 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \end{pmatrix}.$$

In accordance with the annual statement of the company, the values of income under the transition from one state to another state are determined by the following coefficients

$$b_{12} = 2,\ b_{13} = 4,\ b_{21} = 4,\ b_{23} = 4,\ b_{31} = 6,\ b_{32} = 4.$$

Then, given of the initial values $E^{(1)}\{x_0\} = (0,0,0)^T$ and $p(0) = (1,0,0)^T$, for the first moment we get

$$E^{(1)}\{x_1\} = \begin{pmatrix} 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix},$$

further,

$$E^{(1)}\{x_2\} = \begin{pmatrix} 0 & 0,5 & 0,5 \\ 0,5 & 0 & 0,5 \\ 0,5 & 0,5 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 2 \\ 3 \end{pmatrix} + \begin{pmatrix} 0 & 1 & 2 \\ 2 & 0 & 2 \\ 3 & 2 & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0,5 \\ 0,5 \end{pmatrix} = \begin{pmatrix} 4 \\ 2,5 \\ 2 \end{pmatrix},$$

and so on. By using a numerical method, it can be obtained the expected value income, this is approximately $4,33$ with dispersion $1,11$.

Remark 1. *Processes described at system* (11) *can be controlled introducing a control function* U_n.

Let us develop the idea mentioned in the remark above. On a bounded area G, we consider the following stochastic dynamic system with random coefficients

$$x_{n+1} = A(\xi_{n+1}, \xi_n)x_n + B(\xi_{n+1}, \xi_n)U_n, \quad n = 1, 2, \dots \tag{15}$$

where the vectors U_n belong to the set of control U (see [21]). On the space $C^1(G \times U)$ is defined the functional

$$J = E\left\{\sum_{n=0}^{\infty} x_n^* Q(\xi_n)x_n + U_n^* L(\xi_n)U_n\right\}, \tag{16}$$

where the matrices Q, L with Markov elements are symmetric and positive definite. The funktional J is called the quality criterion of control vectors U_n. The control function U_n in the form

$$U_n = S(\xi_n)x_n, \quad n = 1, 2, \dots, \tag{17}$$

which minimizes the quality criterion (16) with respect to the equation (11) is called the optimal control.

We denote $S(\theta_k) = S_k$, $Q(\theta_k) = Q_k$, $L(\theta_k) = L_k$, $k = 1, 2, \dots, q$. If we use the method developed in [21], then we obtain the next result in the form of the following theorem.

Theorem 2. *Let there exist the optimal control* (17), *that minimizes the quality criterion* (16) *with respect to the equation* (11). *Then the matrices* S_k *are determined by the system*

$$S_k = -\left(L_k + \sum_{s=1}^{q} \pi_{sk} B_{sk}^{*} C_s B_{sk}\right)^{-1} \cdot \sum_{s=1}^{q} \pi_{sk} B_{sk}^{*} C_s A_{sk} \qquad (18)$$

where the matrices C_s, $k = 1, 2, \dots, q$ *satisfy the system of matrix equations*

$$\begin{aligned} C_k = Q_k &+ \sum_{s=1}^{q} \pi_{sk} A_{sk}^{*} C_s A_{sk} \\ &+ \sum_{s=1}^{q} \pi_{sk} A_{sk}^{*} C_s B_{sk} \left(L_k + \sum_{s=1}^{q} \pi_{sk} B_{sk}^{*} C_s B_{sk}\right)^{-1} \sum_{s=1}^{q} \pi_{sk} B_{sk}^{*} C_s A_{sk} \end{aligned}$$

with known matrix A_{sk}, B_{sk}.

The Theorem gives the necessary conditions for optimal solution of system (15) and also allows the problem of synthesis of the optimal control to transform to the problem of to determine the matrices S_k in system (18).

4 Model problem 2

We consider inhomogenous difference equation of the form

$$x_{n+1} = x_n + b(\xi_n), \quad n = 1, 2, \dots, \qquad (19)$$

doplni za podm. describing behavior of the random value x_n, which expresses company income at a moment n.

Assume that a company is managed in accordance with the behavior of Markov chain ξ_n with two possible states θ_1, θ_2 and transition matrix

$$\begin{pmatrix} 1-\lambda & \lambda \\ \lambda & 1-\lambda \end{pmatrix}, \qquad 0 \le \lambda \le 1. \qquad (20)$$

If the random variable is in state θ_1 the company makes a profit, if the random variable is in state θ_2 the company has a loss. Let the value of the company's profit or loss is is expressed as

$$b(\theta_1) = \quad \beta,$$
$$b(\theta_2) = -\beta$$

and initial state is $x_0 = 0$. Then equations (13) take the form

$$\begin{aligned} E_1^{(1)}\{x_{n+1}\} &= (1-\lambda)\left(E_1^{(1)}\{x_n\} + \frac{\beta}{2}\right) + \lambda\left(E_2^{(1)}\{x_n\} - \frac{\beta}{2}\right), \\ E_2^{(1)}\{x_{n+1}\} &= \lambda\left(E_1^{(1)}\{x_n\} + \frac{\beta}{2}\right) + (1-\lambda)\left(E_2^{(1)}\{x_n\} - \frac{\beta}{2}\right). \end{aligned} \tag{21}$$

If we consider the initial values

$$E_1^{(1)}\{x_0\} = E_2^{(1)}\{x_0\} = 0, \tag{22}$$

we obtain

$$\begin{aligned} &E_1^{(1)}\{x_{n+1}\} + E_2^{(1)}\{x_{n+1}\} = E^{(1)}\{x_{n+1}\} = 0, \\ &E_1^{(1)}\{x_{n+1}\} - E_2^{(1)}\{x_{n+1}\} = (1-2\lambda)\left(E_1^{(1)}\{x_n\} - E_2^{(1)}\{x_n\} + \beta\right). \end{aligned}$$

It is easy to see, from the first equation, that $E^{(1)}\{x_{n+1}\} = 0$, this means, company will be left without of expected value net profit for the above conditions.

Moment of the second order $E^{(2)}\{x_{n+1}\}$ for the above conditions can be obtained from equations (14) in the form

$$E^{(2)}\{x_n\} = n\beta^2 + \frac{\beta^2}{2\lambda^2}\left(n(1-2\lambda) - (n+1)(1-2\lambda)^2 + (1-2\lambda)^{n+2}\right) \tag{23}$$

Thence, for example, if $\lambda = 0,5$, the moment of the second order is $E^{(2)}\{x_n\} = n\beta^2$, this means, company's net profit varies according to the rule of three sigma in the interval

$$-3\beta\sqrt{n} \le x_n \le 3\beta\sqrt{n}.$$

References

[1] V. V. Anisimov: Random Processes with dicrete components, *Vyshcha shkola, Kiev*, 1988, in Russian.

[2] K. J. Åström: Introduction to stochastic control theory, *Elsevier*, 1970.

[3] E. A. Barbashyn: Lyapunov functions, *Nauka, Moscow*, 1970, in Russian.

[4] I. A. Dzhalladova: Optimization of Stochastic System, *KNEU, Kiev*, 2005, in Russian.

[5] G. Felmer, A. Shid: Vvedenie v stokhasticheskie finansy. Diskretnoe vremja, *MTsMNO, Moscow*, 2008, in Russian.

[6] I.I. Gikhman, A.V. Skorokhod: *Controlled Stochastic Processes*, New York - Heidelberg - Berlin: Springer-Verlag, 1979.

[7] R. Z. Hasminski: Stochastic Stability of Differential Equations, *Sijthoff & Noordhoff, Alphen aan den Rijn*, 1980.

[8] S. M. Hrisanov: Moment Systems Group, *Ukr. Math. Journal*, (1981), 787-792, in Russian.

[9] 8. N. Ikeda, S. Watanabe: Stochastic differential equations and diffusion processes, *North Holland*, 1989.

[10] J. Jacod, A. N. Shiryaev: Limit theorems for stochastic processes, *Springer-Verlag, New York*, 1987.

[11] V. K. Jasinskiy, E. V. Jasinskiy: Problem of stability and stabilization of dynamic systems with finite aftereffect, *TVIMS, Kiev*, (2005).

[12] I. Ya. Katz: Lyapunov Function Method in Stability and Stabilization Problems for Random-Structure Systems *Izd. Ural. Gos. Akad.*, 1998, in Russian.

[13] I. Ya. Katz, N. N. Krasovskii: On Stability of Systems with Random Parameters, *Prikl. Mat. Mekh.*, T. 24, No. 5, 809-823 (1960), 809-823, in Russian.

[14] A. N. Kolmogorov: Selected Works of A.N. Kolmogorov(Probability theory and mathematical statistics), *Springer, New York*, 1992.

[15] V. S. Korolyuk, W. Limnios: Stochastic Systems in Merging Phase Space, *London: Word Scientific*, 2006.

[16] V. S. Korolyuk, V. V. Korolyuk: Stochastic Models of Systems, *Naukova Dumka, Kiev*, 1989, in Russian.

[17] D. H. Korenivskyi: The Destabilizing Effect of Parametric White Noise in Continuous and Discrete Dynamical Systems, *Akademperiodika, Kiev*, 2008, in Russian.

[18] A. M. Lyapunov: General Problem of the Stability of Motion, *Tayor & Francis in London, Washington, DC*, 1992.

[19] B. Oksendal: Stochastic Differential Equations, *Springer-Verlag, Berlin*, 2000.

[20] V. S. Pugachev: Stochastic Systems: Theory and Applications, *World Scientific Publishing Co. Pte. Ltd.*, 2001.

[21] M. Růžičková, I.A. Dzhalladova: The optimization of Solutions of the Dynamic Systems with Random Structure, *Abstr. Appl. Anal.* **2011**, Article ID 486714, 18 pages.

[22] A. M. Samoilenko, N. A. Perestyuk: Impulsive Differential Equations, *Vyshcha Shkola, Kiev*, 1987, in Russian.

[23] A. A. Sveshnikov: Applied Methods of the Theory of Random Functions, *Pergamon Press, Oxford, New York*, 1966.

[24] K.G. Valeev, I. A. Dzhalladova: Optimization of Random Process, *KNEU, Kiev*, 2006, in Russian.

[25] V. I. Zubov: Methods of A. M. Lyapunov and their application, *P. Noordhoff, Groningen*, 1964.

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