УДК 519.925.51

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Про моделювання динаміки портфеля акцій

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On modeling the dynamics of portfolio

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Розглядається задача про побудову математичної моделі оптимального інвестиційного портфеля акцій.
Ключові слова: математична модель, оптимальне інвестування, портфель акцій.

Mathematical models of the dynamics of the formation of the market value per share and stock portfolio are constructed in a class of ordinary differential equations. Such models are needed to build the trajectories of the market price of shares and the optimal portfolio diversification. Initially, procedure is based on the method of variation of arbitrary constants and enables one to find an analytical solution of the corresponding equation. Model of the dynamics of the portfolio is constructed based on the mathematical model of the market value per share. This model is used because of its diversification and for building the initial values of the optimal vector of shares of different types in the portfolio. Mathematical set theory and methods of optimal control theory are applied to achieve that. The problem of partitioning of investment interval on subintervals is studied. On each of them the problem of constructing an optimal investment portfolio is solved taking into account constraints on the control.
Key Words: mathematical model, optimal investment, stock portfolio

Article presented by professor Garashchenko F.G.

1. Introduction

To solve the problem of optimal investment securities following methods are used:

- H. Markowitz approach, using a valid and effective set, investor indifference curves;
- Splitting two-kriterial investment portfolio optimization problem into two one-kriterial;
- Methods of technical analysis;
- Methods of fundamental analysis.

2. Results and Discussion.

Let's consider a mathematical model of the market value of the investment portfolio and build appropriate analytical trajectory. At the time interval $t \in [t_0, T]$ equation describing the profitability of the portfolio shares r_p, has the form

$$r_p(t) = \sum_i x_i(t) r_i(t) , \quad (1)$$

where x_i – proportion of shares and type-portfolio; r_i – expected return on equity and-order type. Having differentiated both parts of (1), we get

$$\frac{dr_p(t)}{dt} = \sum_i (r_i(t)\frac{dx_i(t)}{dt} + x_i(t)\frac{dr_i(t)}{dt}) . \quad (2)$$

Consider normalizing expressions that the content is inverse response time of the system to change the dynamics of expected profitability and structure of portfolio

$$\sum_i \frac{\dot{r}_i(t)}{r_i(t)} = 1, \ \sum_i \frac{\dot{x}_i(t)}{x_i(t)} = 1, \ t \in [t_0, t_1] .$$

Given that there is a property

for $i \neq j$ and $\sum_j b_j = 1$

$$\sum_i a_i b_i = \sum_i a_i + \sum_i \sum_j a_i b_j .$$

In this way occurring ratio

$$\sum_i x_i(t) r_i(t) \frac{f_i}{r_i(t)} = \sum_i x_i(t) r_i(t) - \sum_i \sum_j x_i(t) \times r_i(t) \frac{f_j}{r_j(t)},$$

$$\sum_i \frac{x_i(t) r_i(t)}{x_i(t)} \frac{dx_i(t)}{dt} = \sum_i x_i(t) r_i(t) - \sum_i \sum_j x_i(t) \times r_i(t) \frac{dx_j(t)}{dt} \frac{1}{x_j(t)}.$$

The dynamic model of the market value of the stock portfolio will have the following form

$$\frac{dr_p(t)}{dt} = 2r_p(t) - \sum_i \sum_j x_i(t) r_i(t) \left(\frac{f_j}{r_j(t)} \times \frac{dx_j(t)}{dt} \frac{1}{x_j(t)}\right). \qquad (3)$$

This is a first order linear equation. Applying the method of variation of arbitrary constants and getting its general solution

$$r_p(t) = -\ell^{2t} \int \ell^{-2t} \sum_i \sum_j x_i(t) r_i(t) \left(\frac{f_j}{r_j(t)} \times \frac{dx_j(t)}{dt} \frac{1}{x_j(t)}\right) dt, \qquad (4)$$

where

$$f_j(t) = (\alpha_1 SM_{ind}(t) + \alpha_2 I(t)) r_j(t) + \sum_{i=1}^{N} \beta_{ij} r_j(t).$$

In latter ratio the assumptions made phenomenon that describes the dynamics of the formation of the market value of the portfolio of risky securities. A more detailed analysis indicates two important properties that characterize the market value of the portfolio: dynamics depends on the dynamics of both the expected profitability of shares and changes in the structure of the portfolio. In relation (3) f_j is the right-hand side of the differential equation that describes the dynamics of the formation of the market value of the shares [2]. Its structure and content corresponds to a market model of Sharpe [1].

We were able to construct a sequence paths at selected time intervals, allowing you to create the initial structure of the investment portfolio.

However, this approach can not effectively influence the structure of investment and not fully using the mathematical description of the properties listed in mathematical models (4).

Among other, it contains a factor $\frac{dx_j(t)}{dt} * \frac{1}{x_j(t)}$, that describes the possible changes in the structure of investment.

To simplify further calculations mathematical model (4) is presented in more general terms

$$\dot{r}_p = f^p(r_p, x_i, \dot{x}_i, r_i, \dot{r}_i), \ i = \overline{1, l} \qquad (5)$$

When solving optimal control problem (1), (2) we use the method of successive approximations. As a phase state we consider the market value of equity portfolio r_p and as the control parameters in accordance with the model - vector

$$u(t) = (x_i(t), r_i(t), \frac{dx_i(t)}{dt}, \frac{dr_i(t)}{dt}). \qquad (6)$$

We consider a mathematical model of the dynamics of formation of the market value of the investment portfolio (2); the desired level of the market value of the portfolio at the time T, $r_p(T) = r_P^T$; time interval $t \in [t_0, T]$; control constraints at each time point $x(t) \in U(t)$; quality criterion

$$J(x(t)) = \int_{t_0}^{T} (r_p(t) - r_p(T))^2 dt + \Phi(r_p(t_0)) \to \min_{x(t) \in U(t)} .$$

Here $\Phi(r_p(t_0))$ – given function.

We need to identify $x(t_0)$ and, respectively, $r_p(t_0)$.

At the same time, we recall that the vector x describes the proportion of different types of shares in the portfolio, $x = (x_1, x_2, ..., x_n)$. Here n - number of shares in the investment portfolio types.

To solve the optimal control problem stated above with one fixed end of the trajectory and fixed time using the maximum principle. The result is the ability to identify $x(t_0)$, and on its basis $r_p(t_0)$. The Hamilton function has the form

$$H(r_p(t),\psi(t),t,x(t)) = -(r_p(t)-r_p(T))^2 + \psi(t) * f(r_p, x_i, \dot{x}_i, r_i, \dot{r}_i).$$

A necessary condition for its optimality is

$$\frac{\partial(-(r_p(t)-r_p(T))^2 + \psi(t) * f(r_p, x_i, \dot{x}_i, r_i, \dot{r}_i))}{\partial x_i} = 0,$$

$$i = \overline{1,n}. \quad (7)$$

A solution of equation (3) is the portfolio management function $x^*(\psi(t), r_p(t), t)$.

We construct a conjugated system in the form of

$$\dot{\psi}(t) = -\frac{\partial H(r_p(t),\psi(t),t,x(t))}{\partial r_p},$$

$$\dot{\psi}(t) = 2(r_p(t)-r_p(T)) - \psi(t) * f'_{r_p}(r_p, x_i, \dot{x}_i, r_i, \dot{r}_i).$$

We formulate the transversality condition at the left end of the trajectory

$$\psi(t_0) = -\frac{\partial\Phi(r_p(t_0))}{\partial r_p}.$$

A boundary value problem the maximum principle has the form

$$\begin{cases} \dot{\psi}(t) = 2(r_p(t)-r_p(T)) - \psi(t) * f'_{r_p}((r_p, x_i, \dot{x}_i, r_i, \dot{r}_i) \\ \dot{r}_p(t) = f(r_p, x_i, \dot{x}_i, r_i, \dot{r}) \end{cases}$$

under conditions $\psi(t_0) = \psi_0$, $r_p(T) = r_p^T$.

The right side of the equations of the model depends on the parameters $\alpha_1, \alpha_2 \in A$, where A – a closed bounded set of model parameters. For the correct mathematical formulation of the problem we formulate additional equations

$$\frac{\partial H(r_p(t),\psi(t),t,x(t,\alpha)}{\partial\alpha_j} = 0, \quad j = 1,2.$$

Let's use the statement of the theorem for the optimal control problem with one fixed end of the trajectory and fixed time.

We decide to built a system of ordinary differential equations that determine the function $r_p(t)$, $\psi(t)$, which, when $x^*(\psi(t), r_p(t), t)$ is substituted in the decision, will determine the optimal structure of the investment portfolio in selected time interval.

The analysis algorithm described above makes it possible to note some shortcomings, the most important of which is the parametrically defined objective function and significant limitations on the functions of the control.

While in the classical optimal control theory every part of the optimal trajectory is optimal (or "improvement" of the path leads to "improvement" of the entire path), in a problem with these parameters does not hold true.

Since the optimum value $\alpha^* = (\alpha_1^*, \alpha_2^*)$ for the entire trajectory and its parts cannot be the same, that is, if $x^*(t)$, α^* it's possible to obtain a solution formulated optimal control problem in the interval $t \in [t_0, T]$, at the lower segment due to parameter changes it may be possible to "improve" $x(t)$.

We construct a procedure for determining the optimal stock portfolio structure by constructing a sequence of optimal controls in the sub-intervals, starting with the last, where the value of the objective function in a finite time is known.

Divide the time interval $t \in [t_0, T]$ into subintervals that meet the time when the decision to is optimize the portfolio structure.

For any of the k – sub-slot during $t \in [t_{k-1}, t_k]$ formulation of the problem in a continuous form can be written as

$$\frac{dr_p(t)}{dt} = f(r_p(t), x(t), t), \; t_{k-1} \le t \le t_k, \quad (8)$$

$$r_p(t_k) = r_p^k, \quad x(t_{k-1}) \in U(t_{k-1}), \quad (9)$$

Where $U(t_{k-1})$ – closed bounded set of admissible controls for moment t_{k-1}

$$I(x(t_{k-1}), t_{k-1}) = \min_{x(t_{k-1}) \in U(t_{k-1})} \left(\int_{t_{k-1}}^{t_k} f^0(r_p(t), x(t), t) dt + \Phi(r_p(t_k)) \right). \quad (10)$$

Function $\Phi(r_p(t_k))$ is known.

Statement of the problem (8) - (10) is quite common among the unknown comprising time t_{k-1} and, accordingly, $r_p(t_{k-1})$ - expected market value of the investment portfolio at this time.

Another way to the formulation and solution of the problem of optimization of the investment structure of the portfolio is to apply formulation of two – kriterial problem of H. Markowitz.

For k – interval mathematical model has the following form:

$$r_p(t_{k-1}) = r_p^{k-1} = \sum_{j=1}^{l} x_j(t_{k-1}) r_j(t_{k-1}) \qquad (11)$$

and the challenge is to maximize the expected market value of the investment portfolio. Dynamics of formation of the market value of one share is described by the corresponding model that is detailed in [1] and has the following form:

$$\frac{dr_j(t)}{dt} = (\alpha_1 SM_{ind}(t) + \alpha_2 I(t)) r_j(t) + \sum_{i=1}^{N} \beta_{ij} r_j(t),$$
$$j = \overline{1,n}.$$

It is necessary to define the control function $x(t)$, which will allow the optimal transition from point r_{p_k} to point $r_{p_{k-1}}$.

Criterion for determining the optimal market value of the portfolio to the time t_{k-1} is written in the form

$$\sum_{j=1}^{n} x_j(t_{k-1}) r_j(t_{k-1}) \to \max_{x(t) \in U(t)}. \qquad (12)$$

Method of successive approximations is used for solving the optimal control problem (8), (9). The peculiarity of the application and the method of successive approximations is that the function of controls $x(t)$, $t \in [t_1, t_{k-1}]$ at each time overlaps sufficiently restrictive

$x(t_i) \in U(t_i)$,

where $U(t_i)$ – the set of admissible controls, which is characterized by the presence of the required amount of the respective type of securities and properties in the market.

Application of the method of successive approximations makes it possible to construct a sequence of controls for optimal transition from point $r_p(t_k)$ to point $r_p(t_{k-1})$. Applying steps of the algorithm at the time t_0, we obtain the optimal structure of the initial portfolio.

Thus, the solutions of (11) - (12) make it possible to build a consistent set of paths that enable the investor to form an optimal portfolio of stocks.

3. Acknowledgements.
The formulation of the problem and built algorithm greatly expand the possibilities of investing, as in every moment of diversification it is possible to construct a set of alternatives that are equivalent in terms of quality of selected criteria. The investor, as in the classical approach, Mr. Markowitz has the ability to take into account when deciding additional factors that arise in the course of practical investment and are associated with the peculiarities of the system dynamics modeling. Mathematical portfolio diversification procedure makes it possible for the above-mentioned models of the dynamics of the market value of a share and the stock portfolio to solve the problem of constructing an initial portfolio with a known value of its expected market value at the selected future date.

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Надійшла до редколегії 10.03.2016 р.