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On representations of the semigroups $S(I,J)$ with acyclic quiver

In this paper we study representations of semigroups (over a field k) generated by idempotents with partial null multiplication in the case when the corresponding quiver has not oriented cycles.

We will use the definitions, notions and conventions of [1]. The main of them will be repeated.

Let I be a finite set without 0 and J a subset of $I \times I$ without elements of the form (i,i). We define $S(I,J)$ to be the semigroup with generators e_i, where i runs through all elements of $I \cup 0$, and the following relations:

1) $e_0 = 0$;

2) $e_i^2 = e_i$ for every $i \in I$;

3) $e_i e_j = 0$ for every pair $(i,j) \in J$.

The set of the semigroups of the form $S(I,J)$ is denoted by $\mathcal{I}$. We call $S(I,J) \in \mathcal{I}$ *a semigroup generated by idempotents with partial null multiplication.*

For a finite set X and $Y \subseteq X \times X$, we denote by $Q(X,Y)$ the quiver with vertex set X and arrows $a \to b$, $(a,b) \in Y$. We also put

$$\overline{Y} = \{(a,b) \in (X \times X) \setminus Y \mid a \neq b\}.$$

Throughout, k denotes a field.

Let S be a semigroup. *A matrix representation of S (of degree n) over k* is a homomorphism T from S to the multiplicative semigroup of $M_n(k)$. If there is an identity (resp. zero) element $a \in S$, we assume that the matrix $T(a)$ is identity (resp. zero). In terms of vector spaces and linear transformations, *a representation of S over k* is a homomorphism φ from S to the multiplicative semigroup of the algebra End_kU with U being a finite-dimensional vector space. Two representation $\varphi : S \to End_kU$ and $\varphi' : S \to End_kU'$ are called *equivalent* if there is a linear map $\sigma : U \to U'$ such that $\varphi\sigma = \varphi'$.

A representation $\varphi : S \to End_kU$ of S is also denoted by (U, φ). By the dimension of (U, φ) one means the dimension of U. The representations of S form a category which will be denoted by $\mathrm{rep}_k\, S$ (it has as morphisms from (U,φ) to (U',φ) the maps σ such that $\varphi\sigma = \varphi'$).

In this paper we study representations of semigroups $S(I,J)$ over k in the case when the quiver $Q(I,\overline{J})$ is acyclic, i.e. has not oriented cycles.

Recall the notion of representations of a quiver [2].

Let $Q = (Q_0, Q_1)$ be a finite quiver with the set Q_0 of its vertices and the set Q_1 of its arrows $\alpha : x \to y$.

A representation of the quiver $Q = (Q_0, Q_1)$ over a field k is given by a pair $R = (V, \gamma)$ formed by a collection

$$V = \{V_x \mid x \in Q_0\}$$

of vector spaces V_x and a collection

$$\gamma = \{\gamma_\alpha \mid \alpha : x \to y \text{ runs through } Q_1\}$$

of linear maps $\gamma_\alpha : V_x \to V_y$. A morphism from $R = (V, \gamma)$ to $R' = (V', \gamma')$ is a collection $\lambda = \{\lambda_x \mid x \in Q_0\}$ of linear maps

$\lambda_x : V_x \to V'_x$, such that $\gamma_\alpha \lambda_y = \lambda_x {\gamma'}_\alpha$ for any arrow $\alpha : x \to y$. The category of representations of $Q = (Q_0, Q_1)$ will be denoted by $\operatorname{rep}_k Q$.

A linear map α of $U = U_1 \oplus \dots U_p$ into $V = V_1 \oplus \dots V_q$ is identified with the matrix (α_{ij}), $i = 1, \dots, p$, $j = 1, \dots, q$, where $\alpha_{ij} : U_i \to V_j$ are the induced linear maps.

Let $S = S(I, J)$ with $I = \{1, 2, \dots, m\}$. Define the functor

$$F = F(I, J) : \operatorname{rep}_k Q(I, \overline{J}) \to \operatorname{rep}_k S(I, J)$$

as follows. $F = F(I, J)$ assigns to an object $(V, \gamma) \in \operatorname{rep}_k Q(I, \overline{J})$ the object $(V', \gamma') \in \operatorname{rep}_k S(I, J)$, where

$$V' = \oplus_{i \in I} V_i,$$

$(\gamma'(e_i))_{jj} = \mathbf{1}_{V_j}$ if $i = j$, $(\gamma'(e_i))_{ij} = \gamma_{ij}$ if $(i, j) \in \overline{J}$, and $(\gamma'(e_i))_{js} = 0$ in all other cases. F assigns to a morphism λ of $\operatorname{rep}_k Q(I, \overline{J})$ the morphism $\oplus_{i \in I} \lambda_i$ of $\operatorname{rep}_k S(I, J)$.

In [1] the first author proved that the functor $F = F(I, J)$ is full and faithful (see the only theorem).

From this theorem it follows that a semigroup $S(I, J)$ is wild if so is the quiver $Q(I, \overline{J})$ (the general definitions of tame and wild classification problems are given in [3].

In this paper we prove the following theorem.

Theorem 1. *Let $S(I, J)$ be a semigroup from $\mathcal{I}$ such that the quiver $Q(I, \overline{J})$ is acyclic.*

Then each object of the category $\operatorname{rep}_k S(I, J)$ is isomorphic to an object of the form $XF(I, J) \oplus (W, 0)$, where $X \in \operatorname{rep}_k Q(I, \overline{J})$ and W is a vector space of dimension $d \geq 0$.

Proof. For simplicity, the quiver $Q(I, \overline{J})$ is designated by

$$Q = (Q_0, Q_1).$$

We use the induction on m; the case $m = 0, 1$ are trivial.

Now let $m > 1$ and $R = (U, \varphi)$ be a representation of $S(I, J)$. Fix $s \in Q_0$ such that there is no arrow $i \to s$; we will assume (without loss of generality) that $s = m$. Consider the subsemigroup S'

of S generated by e_i, $i \in I' \cup 0$, where $I' = \{1, \dots, m-1\}$. Then $S' = S(I', J')$ with

$$J' = \{(i,j) \in I \times I \,|\, i, j \in I'\},$$

and $Q' = Q(I', \overline{J'})$ is the full subquiver of Q with vertex set $Q'_0 = I'$.

Denote by $R' = (U, \varphi')$ the restriction of R to S' ($\varphi'(x) = \varphi(x)$ for any $x \in S'$). It follows by induction that

$$R' \cong \overline{R'} = X'F(I', J') \oplus (W', 0),$$

where X' is a representation of the quiver $Q(I', \overline{J'})$.

Let $\overline{R'} = (\overline{U}, \overline{\varphi'})$ and $X' = (V', \gamma')$ with $V' = \{V'_i \,|\, i \in Q'_0\}$ and

$$\gamma' = \{\gamma'_\alpha \,|\, \alpha : i \to j \text{ runs through}\, Q'_1\}.$$

Since $R' \cong \overline{R'}$, there exists a linear map

$$\sigma : U \to \overline{U} = V'_1 \oplus V'_2 \oplus \dots \oplus V'_{m-1} \oplus W'$$

such that $\varphi'\sigma = \overline{\varphi'}$. Then the representation $R = (U, \varphi)$ is equivalent to the representation $\overline{R} = (\overline{U}, \overline{\varphi})$, where $\overline{\varphi}(e_i) = \overline{\varphi'}(e_i)$ for any $i = 1, \dots, m-1$ and $\overline{\varphi}(e_m) = \varphi(e_m)\sigma$ (since, for $i \neq m$, $\overline{\varphi'}(e_i) = \varphi'(e_i)\sigma = \varphi(e_i)\sigma$, and so $\overline{\varphi}(x) = \varphi(x)\sigma$ for each $x \in S$).

We consider the representation $\overline{R} = (\overline{U}, \overline{\varphi})$ in more detail. Put $V_m = W'$ and consider $\overline{\varphi}$ as a matrix, taking into account that

$$\overline{U} = V_1 \oplus V_2 \oplus \dots \oplus V_{m-1} \oplus V_m.$$

For $(p, q) \in J$, we denote by $[p, q, i, j]$ the scalar equality

$$[\overline{\varphi}(e_p)\overline{\varphi}(e_q)]_{ij} = 0,$$

induced by the (matrix) equality $\overline{\varphi}(e_p)\overline{\varphi}(e_q) = 0$ (the last equation holds since $e_p e_q = 0$ in $S(I, J)$). It follows from $[m, q, i, q]$ (for any fixed $q \neq m$) that $(\overline{\varphi}(e_m))_{iq} = 0$, and consequently $(\overline{\varphi}(e_m))_{ij} = 0$ for any $(i, j) \in I \times I'$.

We first consider two special cases:

a) $\overline{\varphi}_{mm} = 0$;

b) $\overline{\varphi}_{mm} = \mathbf{1} = \mathbf{1_{V_m}}$.

In case a) $(\overline{\varphi})^2 = \overline{\varphi}$ implies $\overline{\varphi} = 0$ and so

$$\overline{R} = XF(I,J) \oplus (W,0)$$

with $X = (V,\gamma)$, where $V = \{V'_1 \dots, V'_{m-1}, 0\}$, $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = 0$ for $\alpha \notin Q'_1$ and $W = W'$.

In case b) an equality $[p,m,p,m]$ for $(p,m) \notin \overline{J}$ implies

$$(\overline{\varphi})_{pm} = 0$$

and so

$$\overline{R} = XF(I,J) \oplus (W,0)$$

with $X = (V,\gamma)$, where $V = \{V'_1 \dots, V'_{m-1}, 0\}$, $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = 0$ for $\alpha \notin Q'_1$ and $W = W'$.

Now we consider the general case. Since $(\overline{\varphi}_{mm})^2 = \overline{\varphi}_{mm}$, there is an invertible map $\nu = (\nu_1,\nu_2) : V_m \to W_1 \oplus W_2$ such that

$$\overline{\varphi}_{mm}(\nu_1,\nu_2) = (\nu_1,\nu_2)\begin{pmatrix} \mathbf{1} & 0 \\ 0 & 0 \end{pmatrix},$$

where $\mathbf{1} = \mathbf{1}_{W_1}$. Then the representation $\overline{R'} = (\overline{U}, \overline{\varphi'})$ is isomorphic to the the representation $\widehat{R'} = (\widehat{U}, \widehat{\varphi}')$, where

$$\widehat{U} = \widehat{U}_1 \oplus \widehat{U_2} \oplus \ldots \oplus \widehat{U}_{m+1}$$

with $\widehat{U}_i = V_i$ for $i = 1, \dots m-1$, $\widehat{U}_m = W_1$, $\widehat{U}_{m+1} = W_2$, and $\widehat{\varphi}'(e_i) = \overline{\varphi'}(e_i)$ for $i = 1, \dots m-1$, $(\widehat{\varphi}'(e_m))_{ij} = (\overline{\varphi'}(e_m))_{ij}$ for $(i,j) \in I' \times I'$, $(\widehat{\varphi}'(e_m))_{ij} = 0$ for $i = m, m+1, j \in I'$, $(\widehat{\varphi}'(e_m))_{m,mj} = \mathbf{1} = \mathbf{1}_{W_1}$, $(\widehat{\varphi}'(e_m))_{m,m+1} = 0$, $(\widehat{\varphi}'(e_m))_{m+1,m} = 0$, $(\widehat{\varphi}'(e_m))_{m+1,m+1} = 0$ (for instance, one can take the isomorphism $\beta : \widehat{R'} \to \overline{R'}$ with $\widehat{\varphi}'(e_i) = \mu^{-1}\overline{R'}\mu$, where $\mu = \mathbf{1}_{U_1} \oplus \ldots \oplus \mathbf{1}_{U_{m-1}} \oplus \nu$.

From $(\widehat{\varphi}'(e_i))^2 = \widehat{\varphi}'(e_i)$ it follows that $\widehat{\varphi}'(e_m))_{i,m+1} = 0$ for any $i \in I'$ (see the partial case a)); (then $\widehat{\varphi}'(e_m))_{i,m+1} = 0$ for any $i = 1, \dots, m+1$). From the scalar equalities $[p,m,p,m]$ for $(p,m) \notin \overline{J}$ implies $(\widehat{\varphi})_{pm} = 0$ (see the partial case b)). Thus,

$$\overline{R} = (\widehat{U}, \widehat{\varphi}) \cong R = (U, \varphi)$$

has the form $XF(I,J) \oplus (W,0)$, where $X = (V,\gamma)$ with $V = \{\widehat{U}_i \,|\, i \in Q_0\}$, $\gamma = \{\gamma_\alpha \,|\, \alpha : i \to j \text{ runs through } Q_1\}$ with $\gamma_\alpha = \gamma'_\alpha$ for $\alpha \in Q'_1$, $\gamma_\alpha = \widehat{\varphi}(e_m)_{ij}$ for $\alpha \notin Q'_1$ (then $j = m$), and

$$W = \widehat{W_{m+1}},$$

as claimed. $\square$

Let $\mathrm{rep}^\circ_k\, S(I,J)$ denotes the full subcategory of the category $\mathrm{rep}_k\, S(I,J)$ consisting of all objects that have no objects $(W,0)$, with $W \neq 0$, as direct summands.

Then from the Theorem of [1] and Theorem 1 it follows the following statement.

Theorem 2. *Let $S(I,J)$ be as in Theorem 1. Then the functor $F = F(I,J)$, viewed as a functor from $\mathrm{rep}_k\, Q(I,\overline{J})$ to $\mathrm{rep}^\circ_k\, S(I,J)$, is an equivalence of categories.*

From this theorem it follows that a semigroup $S(I,J)$ and the quiver $Q(I,\overline{J}$ have the same representation type (in the case when $Q(I,\overline{J}$ is acyclic).

References

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